Construction of the Absorbing Boundary Conditions for the FDTD Method with Transfer Functions

Jianyi Zhou and Wei Hong

***Abstract*—The absorbing boundary conditions (ABC's) are essential to terminate the computational space when the finite-difference time-domain (FDTD) method is applied to analyze electromagnetic (EM) problems. With the ABC's, the fields on the truncated boundaries are evaluated by the interior fields. In this paper, the relationship between the fields on the terminated boundaries and the interior fields is expressed as the transfer functions in the Z-domain. The proper transfer functions are determined from the radiation condition or the transmission condition. Simplifying these transfer functions into rational functions, we obtain different schemes of the ABC's. In this paper, both the transfer functions and coefficients of the final ABC's are derived and expressed as recurrence formulas for the convenience of programming. This method has the property of simplicity and flexibility. The ABC's obtained by this method show good absorbing performance and numerical stability in the practical applications.**

***Index Terms*—FDTD methods.**

I. Introduction

The finite-difference time-domain (FDTD) method is a full-wave approach for the analysis of electromagnetic (EM) problems [1]. When the FDTD method is used to solve open-structure EM problems such as scattering, antenna, and discontinuity problems, absorbing boundary conditions (ABC's) are required to terminate the meshes.

Investigators have paid much attention to the application and improvement of ABC's during the last several decades. There are two popular ways used to construct the ABC's, one is the usage of outgoing wave equations, another is the employment of nonphysical absorbing media. Engquist and Majda [2] first used the outgoing wave equations to construct ABC's, and Mur [3] improved them and presented the widely used Mur's ABC's. Mur's ABC's can adequately absorb the outgoing waves that are incident perpendicular to the truncated boundaries. However, when the incident angles of the outgoing waves increase, the absorbing performance of the Mur's ABC's rapidly becomes worse. The dispersive boundary conditions developed by Bi [4] and improved by Zhao *et al.* [5] can adequately absorb the outgoing waves incident with several different angles. On the other hand, great development has been made on the usage of nonphysical absorbing media to truncate the FDTD meshes in the last two years. The perfect matched layer (PML) proposed by Berenger [6] can perfectly absorb the scattered waves with the incident angle in a large range.

In this paper, we use the transfer function to construct the ABC's for the first time. Proper transfer functions are determined from the radiation condition or the transmission condition and, by some simple transformation, the FD schemes of the ABC's are obtained. In this paper, we present the transfer functions and coefficients of the final ABC's with recurrence formulas. These recurrence formulas are very convenience for programming. The validity and priority of the method are shown by several numerical examples.

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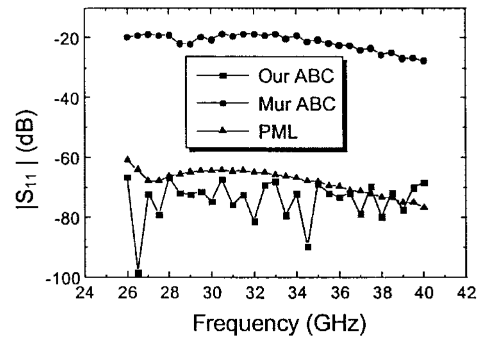

Fig. 1. Reflection caused by different ABC's.

II. Construction of the ABC's

For convenience, we use the transfer functions in the Z-domain to describe the characteristic of the system because it is very easy to transform the transfer functions in the Z-domain into the FD schemes. We assume that the scattered wave propagates along the $-x$-direction with the phase velocity ν, and the truncated boundary is set at $x = 0$. In the Z-domain, the relationship between the boundary fields and interior fields is expressed as

$$E_0(z) = \sum_{k=1}^{p} h_k(z) E_k(z) \tag{1}$$

where $E_0(z)$ is the Z-transformation of the boundary field $E_k(z)$, $k = 1, 2, \cdots, p$ are the Z-transformations of the interior fields at k steps away from the boundary along the x-direction, $h_k(z)$ are the transfer functions, and p is the order of the ABC.

According to the radiation condition or the transmission condition, $E_k(z)$, $k = 1, 2, \cdots, p-1$ must satisfy

$$E_k(z) = z^{-s} E_{k+1}(z) \tag{2}$$

where $s = (\Delta x / \nu \Delta t)$, Δx, Δt are the space and time increments, respectively. Thus, a wave propagates along the $-x$-direction with the phase velocity ν being adequately absorbed. The following linear equation about $h_k(z)$ is obtained:

$$d^p = \sum_{k=1}^{p} d^{p-k} h_k(z) \tag{3}$$

where $d = z^{-s}$.

In practice, the phase velocity ν is determined by the incident angle θ of the scattered wave or the effective dielectric constant ϵ_{reff} of the transmission lines, i.e., $\nu = c/\cos\theta$ or $\nu = c/\sqrt{\epsilon_{\text{reff}}}$, where c denotes the light velocity. To absorb the scattered wave with a different incident angle in a wide-frequency band, p different velocities ν_i, $i = 1, 2, \cdots, p$ are used. Thus, a system of p linear equations is obtained as follows:

$$\begin{bmatrix} d_1^{p-1} & d_1^{p-2} & \cdots & 1 \\ d_2^{p-1} & d_2^{p-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ d_p^{p-1} & d_p^{p-2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} h_1(z) \\ h_2(z) \\ \vdots \\ h_p(z) \end{bmatrix} = \begin{bmatrix} d_1^p \\ d_2^p \\ \vdots \\ d_p^p \end{bmatrix}. \tag{4}$$

Observing (4) carefully, we find that d_i are the p roots of the following polynomial equation:

$$x^p - h_1(z)x^{p-1} - h_2(z)x^{p-2} - \cdots - h_p(z) = 0. \tag{5}$$

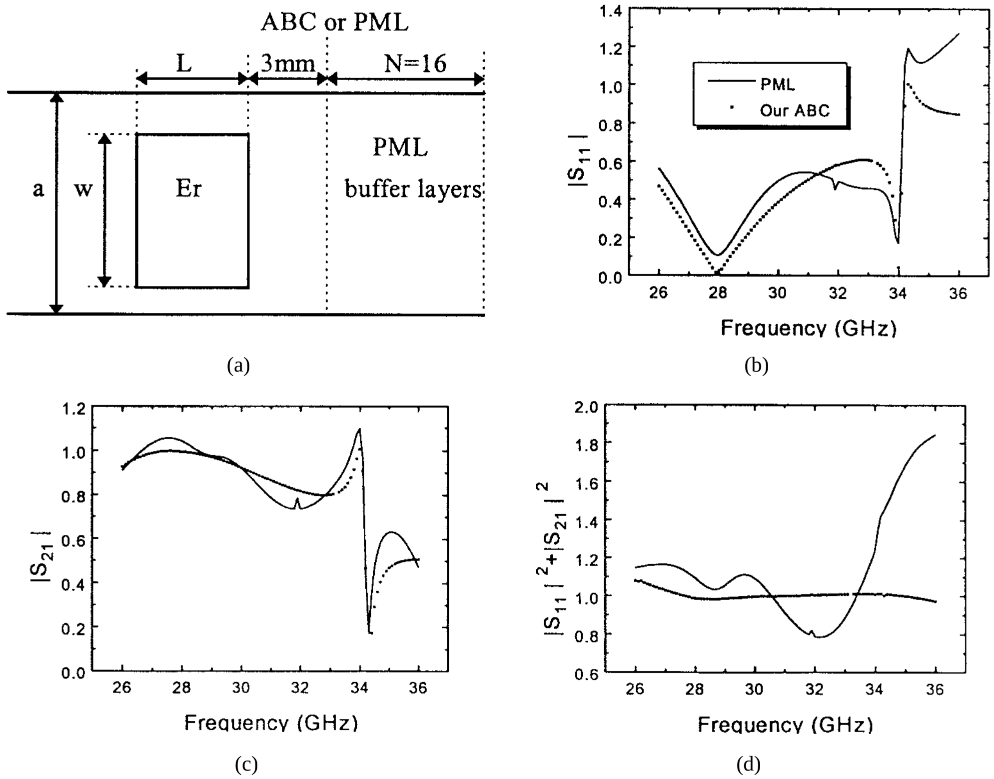

Fig. 2. Configuration and numerical results of a rectangular post in a waveguide. (a) Configuration of the discontinuity. (b) $|S_{11}|$ versus the frequency. (c) $|S_{21}|$ versus the frequency. (d) $|S_{11}|^2 + |S_{21}|^2$ versus the frequency.

On the other hand, (5) can be written as

$$\prod_{i-1}^{p} (x - d_i) = 0. \tag{6}$$

By expanding (6) and comparing the coefficients with (5) yields the transfer functions $h_k(z)$. In fact, the recurrence formulas of the transfer functions $h_k(z)$ can be derived as follows:

$$h_1^{(1)}(z) = d_1 \tag{7}$$

$$h_1^{(p+1)}(z) = h_1^{(p)}(z) + d_{p+1} \tag{8}$$

$$h_k^{(p+1)}(z) = h_k^{(p)}(z) - d_{p+1} h_{k-1}^{(p)}(z), \qquad k = 2, 3, \cdots, p \tag{9}$$

$$h_{p+1}^{(p+1)}(z) = -d_{p+1} h_p^{(p)}(z) \tag{10}$$

where, $h_k^{(p)}$ is the kth transfer function of a pth-order ABC.

In the transfer functions, the factor s is usually not an integer, which means that time delay of a space increment is not an integer time of the time increment, thus, the transfer functions cannot directly be changed into difference schemes. In practical applications, proper simplification is required to transform these transfer functions into rational functions. Some examples are as follows:

$$z^{-s} \approx 1 + s(z^{-1} - 1) = (1 - s) + sz^{-1} \tag{11}$$

$$z^{-s} = \frac{1}{z^s} \approx \frac{1}{1 - s(z^{-1} - 1)} = \frac{1}{(1 + s) - sz^{-1}} \tag{12}$$

$$z^{-s} = \frac{z - \frac{1+s}{2}}{z - \frac{1-s}{2}} \approx \frac{1 + \frac{1+s}{2}(z^{-1} - 1)}{1 + \frac{1-s}{2}(z^{-1} - 1)} = \frac{(1 - s) + (1 + s)z^{-1}}{(1 + s) + (1 - s)z^{-1}} \tag{13}$$

where the factor z^{-1} is a time-increment delay. In these examples, (12) is identical to the dispersive boundary condition in [5], and (13) is identical to the first-order Mur's or Higdon's ABC's [7]. Though (13) is more accurate than (11) and (12), it tends to be unstable in high-order applications. In this paper, (11) is used. It is stable and simple. The FD scheme of the boundary condition can be expressed as

$$E_0^n = \sum_{k=1}^{p} \sum_{j=0}^{k} c_{k,j}^{(p)} E_k^{n-j} \tag{14}$$

where E_k^{n-j} denotes the electric field at time $(n - j)\delta t$ and $k\delta x$ away from the boundary, $c_{k,j}^{(p)}$ is the coefficient which denotes the coefficient of the item E_k^{n-j} in the pth-order ABC. There is no item E_0^{n-k} in (14), which means there is no direct feedback in the ABC, thus, the ABC's have better stability as the finite-impulse response (FIR) filters. There are recurrence formulas for the coefficients $c_{k,j}^{(p)}$. If we let

$$a_i = (1 - s_i) \tag{15}$$

$$b_i = s_i \tag{16}$$

then we have

$$c_{1,0}^{(1)} = a_1 \tag{17}$$

$$c_{1,1}^{(1)} = b_1 \tag{18}$$

$$c_{1,0}^{(p+1)} = c_{1,0}^{(p)} + a_{p+1} \tag{19}$$

$$c_{1,1}^{(p+1)} = c_{1,1}^{(p)} + b_{p+1} \tag{20}$$

$$c_{k,0}^{(p+1)} = c_{k,0}^{(p)} - a^{p+1} c_{k-1,0}^{(p)}, \qquad k = 2, \cdots, p \tag{21}$$

$$c_{k,j}^{(p+1)} = c_{k,j}^{(p)} - a_{p+1} c_{k-1,j}^{(p)} - b_{p+1} c_{k-1,j-1}^{(p)}, \qquad k = 2, \cdots, p, \quad j = 1, \cdots, k-1 \tag{22}$$

$$c_{k,k}^{(p+1)} = c_{k,k}^{(p)} - b_{p+1} c_{k-1.k-1}^{(p)}, \qquad k = 2, \cdots, p \tag{23}$$

$$c_{p+1,0}^{(p+1)} = -a_{p+1} c_{p,0}^{(p)} \tag{24}$$

$$c_{p+1,j}^{(p+1)} = -a_{p+1} c_{p,j}^{(p)} - b_{p+1} c_{p,j-1}^{(p)}, \qquad j = 1, \cdots, p \tag{25}$$

$$c_{p+1,p+1}^{(p+1)} = -b_{p+1} c^{(p)} p, p. \tag{26}$$

If the scattered wave attenuates along the $-x$-direction with the factor α, then d can be replaced as $d = e^{-\alpha \delta x} z^{-s}$. When high-order ABC's are used, approximate values of ν_i and α_i are adequate. In fact, one may select these values in a range according to the incident angles and attenuation fact of the wave in the frequency band of interest.

III. Numerical Results

In this paper, we give the reflection coefficient caused by a third-order ABC, which is developed when it is used to analyze a rectangular waveguide WR-28. The results are shown in Fig. 1, together with the results obtained when we use the first-order Mur's ABC and 16-layer PML as comparison. The parameters of the ABC were selected as $\nu_1 = 1.2c$, $\nu = 1.4c$, and $\nu_3 = 1.6c$. The type of PML used was PML(16, P, 0.001).

The ABC has been used to calculate the S-parameters of a rectangular dielectric post in a rectangular waveguide WR-28. The cross section of the post is 4 mm $\times$ 2 mm, and the dielectric constant ϵr is 8.2, as shown in Fig. 2(a). Resonance occurred near the frequency $f = 34$ GHz, and very strong high-order modes were excited. In this case, if the truncated plane is set very close to the discontinuity, the ABC should absorb both the dominant and high-order modes, which is demonstrated in [8]. A fifth-order ABC was applied where the phase velocities were selected as $\nu_1 = 1.2c$, $\nu_2 = 1.4c$, $\nu_3 = 1.6c$, $\nu_4 \gg c$, $\nu_5 \gg c$, and the attenuation factors were selected as $\alpha_1 \delta x = 0$, $\alpha_2 \delta x = 0$, $\alpha_3 \delta x = 0$, $\alpha_4 \delta x = 0.02$, $\alpha_5 \delta x = 0.05$. Better results were obtained when the distance between the ABC and discontinuity is 3 mm, as shown in Fig. 2(b)–(d). Since the ordinary PML cannot absorb the high-order (cutoff) modes adequately, it fails to obtain correct results unless the distance is much larger. Our ABC can obtain reasonable results even if the distance is very small, and the computational memory and time can be greatly saved. Since the derivation is not limited two-dimensionally, the ABC can be applied in a three-dimensional (3-D) problems.

IV. Conclusion

In this paper, we use transfer functions to construct ABC's for the first time. Recurrence formulas for the transfer functions and coefficients of the final FD schemes of the ABC's are developed. It is quite simple and convenient to apply the ABC's in the FDTD iteration. A lot of computational time and memory can be saved with the ABC's. Numerical results show the good absorbing performance of the ABC in practical problems.

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Edge-Element Formulation of Three-Dimensional Structures

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***Abstract*—A three-dimensional (3-D) asymmetrical functional is developed and implemented as a hybrid-vector edge-element method. The equivalent frequency-dependent circuit parameters are then extracted from the field solutions. Laboratory measurements and data comparison with previous published results strongly support the newly developed theoretical work.**

***Index Terms*—Edge element, functional.**

I. Introduction

In this paper, we have developed a new functional for general three-dimensional (3-D) guided-wave structures, which need not have completely closed metallic walls. We shall then derive the termination conditions at the planes of incidence and transmittance. Utilizing prior information of the eigenmodes resulting from the evaluation of the two-and-one-half-dimensional edge-element solver [1], the 3-D field solutions are obtained. The frequency-dependent circuit parameters (such as L, C, R, and G) are converted according to relevant equivalent circuits of the structures.

II. Basic Theory

We begin with the vector-wave-propagation equation

$$\nabla \times \frac{1}{\mu_r} \nabla \times \vec{E} - \vec{\vec{\epsilon}}_r k_0^2 \cdot \vec{E} = -j\omega\mu \vec{J} \quad \text{in } V. \tag{1}$$

The boundary conditions for (1) are

$$\begin{cases} \hat{n} \times \vec{E} = \vec{P}, & \text{on } S_1 \\ \dfrac{1}{\mu_r} \hat{n} \times \nabla \times \vec{E} + \gamma_v \hat{n} \times \hat{n} \times \vec{E} = \vec{V}, & \text{on } S_2. \end{cases} \tag{2}$$

In the previous equations, S_1 is the surface where the boundary condition of the first kind applies, S_2 is the surface where the boundary condition of the third kind applies, and $\gamma_v = jk_0 \sqrt{(\epsilon_{rc} - j(\sigma/\omega\epsilon_0)/\mu_{rc})}$, as defined in [1]. In the application of this theory to transmission-line structures and their discontinuities, the field component in the signal-propagation direction is generally nonzero, and the aforementioned boundary conditions are insufficient. On both the incident and transmitted planes, the longitudinal

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